ON A CLASS OF MOTIONS IN MAGNETO-HYDROMECHANICS

(OB ODNOM KLASSE DVIZHENII V MAGNITNOI GIDROMEKHANIKE)

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The equations of magneto-hydromechanics for an ideal medium (i.e. a medium which is free from losses due to Joule heating, viscosity and heat conduction) can be written (see, for example, Ref. [1]) in the form:

$$\operatorname{div}\mathbf{H}=0,\qquad \rho\frac{d}{dt}\left(\frac{\mathbf{H}}{\rho}\right)=(\mathbf{H}\nabla)\mathbf{v}$$
$$\frac{\partial\rho}{\partial t}+\operatorname{div}(\rho\mathbf{v})=0,\qquad \rho\frac{d\mathbf{v}}{dt}+\nabla p^{*}=\frac{1}{4\pi}(\mathbf{H}\nabla)\mathbf{H} \tag{1}$$
$$\frac{ds}{dt}=0,\qquad p^{*}=p+\frac{H^2}{8\pi}$$

Here p is the pressure, ρ the density, s the entropy per unit mass, $\mathbf{v}$ the vector velocity of particles of the gas, $\mathbf{H}$ the vector intensity of the magnetic field, d/dt the derivative with respect to time, following a given particle.

The system of equations (1) has to be supplemented with the equation of state of the medium, which we shall write as:

$$p=f(\rho,\ s) \tag{2}$$

In what follows we shall consider the class of motions which satisfy the following conditions:

$$(\mathbf{H}\nabla)\mathbf{H}=0,\qquad (\mathbf{H}\nabla)\mathbf{v}=0 \tag{3}$$

The physical conditions (3) mean that the vectors $\mathbf{H}$ and $\mathbf{v}$ do not change along lines of magnetic force.

Important cases of this class of motions are the plane and axisymmetric flows (in the general case, unsteady) where the vectors of velocity and magnetic field intensity are perpendicular. Let us note that one-dimensional unsteady flow, which is also a particular case of the class of motions considered here, was studied in the paper [2].

On taking into account the conditions (3), the equations of motion assume the form:

$$\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho\mathbf{v}) = 0, \qquad \rho\frac{d\mathbf{v}}{dt} + \nabla p^* = 0, \qquad \frac{ds}{dt} = 0$$

$$p^* = f(\rho,\ s) + k^2\rho^2, \qquad \mathbf{H} = 2\sqrt{2\pi}\ \mathbf{k}\cdot\rho \tag{4}$$

where $\mathbf{k}$ is a vector function which is constant along the trajectory lines of the flow.

Let us consider the change in the circulation of the velocity vector $\Gamma = \oint \mathbf{v}d\mathbf{e}$ around fluid contours (i.e. contours moving with the gas stream for the class of motions satisfying the equations (4).

$$\frac{d\Gamma}{dt} = \frac{d}{dt}\oint \mathbf{v}d\mathbf{e}' = \oint \frac{d\mathbf{v}}{dt}d\mathbf{e}' \tag{5}$$

Substituting in equation (5) the value of the acceleration $d\mathbf{v}/dt$ from the second equation of system (4) and applying Stokes' theorem, we have:

$$\frac{d\Gamma}{dt} = \iint\limits_S \frac{[\nabla\rho\cdot\nabla p^*]}{\rho^2}\,d\mathbf{S} \tag{6}$$

($\mathbf{S}$ is a surface "stretched" across the contour under consideration). Introducing the expression for p^* given in (4), we obtain:

$$\frac{d\Gamma}{dt} = \iint\limits_S \left(\frac{q}{\rho^2}[\nabla\rho\cdot\nabla s] + 2k[\nabla\rho\cdot\nabla k]\right)d\mathbf{S} \qquad \left(q = \frac{\partial f}{\partial s}\right) \tag{7}$$

Accordingly, if the flow of the class under consideration is isentropic and the quantity k is uniform in space, then the circulation around fluid contours has the property that it is conserved. The property so obtained is analogous with Thomson's theorem for the motions under consideration. It follows from the demonstration of this property that there exist irrotational flows in the class under consideration.

The second equation of system (4) can be transformed into the following:

$$\frac{\partial\mathbf{v}}{\partial t} + \nabla\frac{v^2}{2} + \frac{1}{\rho}\nabla p^* - [\mathbf{v}\operatorname{rot}\mathbf{v}] = 0 \tag{8}$$

Projecting the vector equation (8) upon the direction of the velocity vector $\mathbf{v}$, we have:

$$\left(\frac{\partial\mathbf{v}}{\partial t}\right)_{\mathrm{e}} + \frac{d}{de}\left(\frac{v^2}{2}\right) + \frac{1}{\rho}\frac{dp^*}{de} = 0 \tag{9}$$

Moreover, assuming that the motion is steady and integrating equation (9), whilst bearing in mind the relations (4), we obtain an analog of Bernoulli's equation for the motions under consideration:

$$\frac{v^2}{2} + \int\frac{dp_\cdot}{\rho} + \frac{H^2}{4\pi\rho} = \text{const} = \omega \tag{10}$$

The integral (10) holds good along streamlines, but in the case of isentropic flow and if the quantity k is constant it holds for the whole of space (i.e. in the latter case ω is a universal constant for the entire flow).

Considering now a motion which is unsteady and irrotational and integrating equation (9), we obtain the following analog of Lagrange's equation:

$$\frac{\partial\varphi}{\partial t} + \frac{v^2}{2} + \int\frac{dp}{\rho} + \frac{H^2}{4\pi\rho} = \omega_1(t) \qquad (11$$

where ω_1 is a universal function of time t for a given flow, and ϕ is the velocity potential.

The majority of known methods in ordinary hydromechanics (as, for example, the method of characteristics, the method of linearized flows, and so on) can be carried over without difficulty to the case under consideration.

Motions analogous to Prandtl-Meyer flows and Busemann's conical flows can also be easily obtained as members of the class under consideration. In the isentropic case, if the quantity k is constant throughout the flow, the motions under consideration reduce to isentropic gas dynamics, in which the role of the pressure is assumed by the quantity p^*, and the medium is changed so that the adiabat is expressed according to the formula

$$p^* = f(\rho, s_0) + k^2\rho^2 \qquad (12)$$

In conclusion, we note that these same key simplifications for problems of magneto-hydromechanics can be obtained in the case of motion of a viscous heat-conducting gas, if the conditions (3) are again satisfied.

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